Measuring Coil Q: Formula Derivation

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I wanted to write down the derivation of the formula for the LCR circuit equivalent series resistance in these measurements with all the algebra shown in case this level of detail is useful to someone.

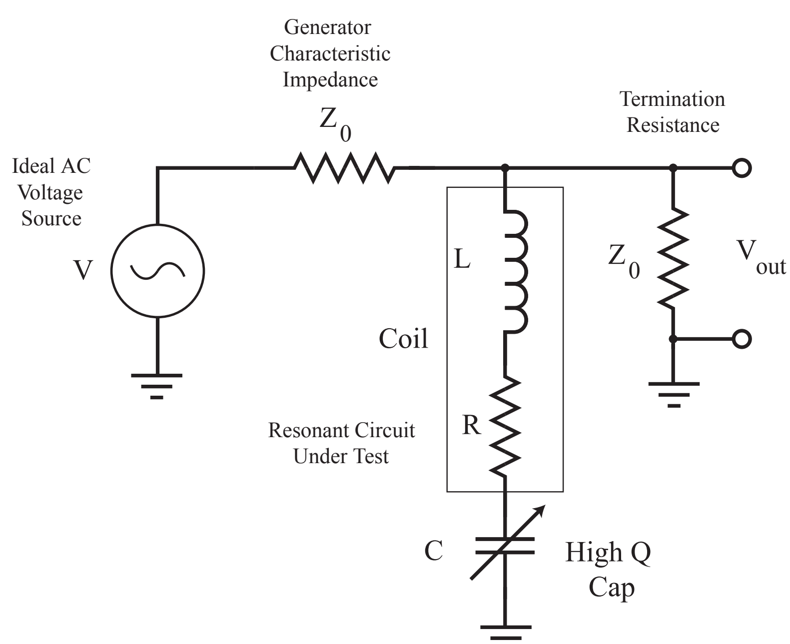

Figure 1: Test setup equivalent circuit. This is drawn assuming the capacitor ESR is negligible compared to the coil, so that R is just the resistance of the coil. Really, the setup measures the total ESR: the coil will dominate but it's easy to get 10% level errors in Q. If it really makes a big difference in your application, you should estimate the capacitor ESR somehow and subtract it out.

If we disconnect the series LCR circuit, we have a voltage divider with equal impedances Z_0, so the output is half the generator voltage. Let's call this the unloaded voltage V_u:

$$V_{out} = V_u = \frac{VZ_0}{2Z_0} = \frac{V}{2} \tag{1}$$

When we reconnect the LCR and resonate it for the minimum V_{out} the inductor and capacitor cancel each other and all that is left is a voltage divider with the output across the parallel combination of R and Z_0. We'll call the output voltage at LCR resonance V_r:

$$V_{out} = V_r = \frac{VR||Z_0}{Z_0 + R||Z_0} \tag{2}$$

Explicitly writing this out:

$$V_r = \frac{V\frac{1}{1/R+1/Z_0}}{Z_0 + \frac{1}{1/R+1/Z_0}} \tag{3}$$

This is ugly. Let's simplify it a bit. Multiply both sides by the right hand side's denominator:

$$V_rZ_0 + V_r\frac{1}{1/R + 1/Z_0} = V\frac{1}{1/R + 1/Z_0} \tag{4}$$

Multiply both sides by $(1/R + 1/Z_0)$:

$$V_rZ_0(1/R + 1/Z_0) + V_r = V \tag{5}$$

or:

$$V_r\frac{Zo}{R} + 2V_r = V \tag{6}$$

Now multiply both sides by R:

$$V_rZo + 2V_rR = VR \tag{7}$$

or:

$$V_r(Z_0 + 2R) = VR \tag{8}$$

Divide by $(Z_0 + 2R)$ to get a simple expression for V_r:

$$V_r = \frac{VR}{Z_0 + 2R} \tag{9}$$

Now take the **ratio** of V_u to V_r. We do this because as long as the receiver is converting the RF voltages to audio voltages in a nice linear fashion, the ratio is something we can measure easily at the receiver **output**. We don't need to know the precise values of the RF voltages at the receiver **input** at any point. The ratio V_u to V_r is the same for the large, easy to measure audio voltages as it is for the tiny, hard to measure μV RF voltages at the receiver input. Generator voltage V cancels, leaving:

$$\frac{V_u}{V_r} = \frac{Z_0 + 2R}{2R} \tag{10}$$

Multiply both sides by $2R$ then subtract $2R$ from both:

$$2R\frac{V_u}{V_r} - 2R = Z_0 \tag{11}$$

or:

$$2R(\frac{V_u}{V_r} - 1) = Z_0 \tag{12}$$

so, finally the resonant circuit ESR R in terms of the measured voltage **ratio** V_u/V_r and the characteristic impedance Z_0 is:

$$R = \frac{Z_0}{2(\frac{V_u}{V_r} - 1)} \tag{13}$$

Since we know R, if we know the inductance L (can be measured directly or from the required resonating capacitance) we can calculate the coil's Q assuming R is due entirely to the coil:

$$Q = \frac{X_L}{R} = \frac{2\pi f L}{R} \tag{14}$$

Substituting in Eq. 13 for R gives:

$$Q = \frac{4\pi f L}{Z_0}\left(\frac{V_u}{V_r} - 1\right) \tag{15}$$

This can be found elsewhere, like in VE2AZX's article in Jan/Feb 2012 QEX and W7ZOI's webpage at http://w7zoi.net/2faces/twofaces.html in terms of the dB attenuation A:

$$A = 20\log_{10}\left(\frac{V_u}{V_r}\right) \tag{16}$$

such that Eq. 15 turns into:

$$Q = \frac{4\pi f L}{Z_0}\left(10^{\frac{A}{20}} - 1\right) \tag{17}$$

Since I am directly measuring the audio voltage from my reciever for the disconnected and resonant conditions, I just use Eq. 15 directly. Another description of all of this by W0XI is available at midnightscience.com, PDF link